Extended Abstracts

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Poisson cohomology: old and new

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Resum *(CAT)*
Les varietats de Poisson tenen naturalment associat un complex de cocadenes, la cohomologia del qual es diu *cohomologia de Poisson*. En aquest treball estenem les tècniques de Guillemin, Miranda i Pires per calcular la cohomologia de Poisson de varietats b^m-Poisson. La cohomologia resultant té dimensió infinita i ve parametritzada per la foliació simplèctica del lloc singular Z. El fet que puguem mesurar la diferència entre la cohomologia de Poisson i la cohomologia de l'algebroide subjacent obre una porta a l'estudi de la cohomologia de Poisson de varietats més generals.

Keywords: *Poisson cohomology, b^m-symplectic geometry, deformation quantization, deformation theory.*

Abstract

Poisson geometry is a vast generalization of the Hamiltonian setting of classical mechanics, described abstractly by symplectic geometry. The deformation of Poisson structures is related to the theory of normal forms and to the problem of their classification. The deformation theory of a Poisson manifold (M, Π) is governed by the cohomology of the Lichnerowicz complex $(\mathfrak{X}^\bullet, \mathrm{d}_\Pi)$, called Poisson cohomology.

For an important class of Poisson structures, the singularities can be encoded in a suitable vector bundle E and analyzed by means of geometric techniques. Such Poisson structures Π are obtained from a symplectic Lie algebroid (E, ρ, ω) by pushforward under the anchor map. These structures were already introduced by Nest and Tsygan [3] in the study of deformation quantization of Poisson manifolds. Their differential and symplectic geometry was thoroughly investigated by Miranda and Scott [2]. A notable instance of this definition is b-symplectic geometry, where $E = {}^b\mathrm{T}M$ is called the b-tangent bundle. Sections of ${}^b\mathrm{T}M$ can be identified with smooth vector fields $X \in \mathfrak{X}(M)$ tangent to an embedded hypersurface $Z \subset M$. A generalization of this setting is b^m-symplectic geometry, where vector fields are assumed to be tangent to Z at least of order m. The work of Scott [4] laid the rudiments of such geometries.

The systematic use of these techniques in the study of Poisson cohomology was pioneered by Guillemin, Miranda and Pires [1] in the case of b-geometry. The complex of sections of ${}^b\mathrm{T}M$ is naturally included in the complex of smooth multi-vector fields and admits a restriction of the operator d_Π. Consequently, the cohomology of this sub-complex, called *b-Poisson cohomology*, is a subspace of the standard Poisson cohomology groups. A comparison lemma due to Mărcuţ and Osorno shows the inclusion morphism induces an isomorphism at the level of cohomology.

In this master thesis we expand on these techniques to answer the following question posed by Alan Weinstein: is the inclusion morphism an isomorphism in b^m-symplectic geometry? We prove the quotient complex $\mathfrak{X}^\bullet_Q(M) = \mathfrak{X}^\bullet(M)/{}^{b^m}\mathfrak{X}^\bullet(M)$, which measures the obstruction to the inclusion being an isomorphism in cohomology, has infinite-dimensional cohomology groups. By showing this complex can be localized to the degeneracy locus $Z \subseteq M$ of Π, we are able to use normal form theory to completely describe the cohomology groups of $\mathfrak{X}^\bullet_Q(M)$ and, using a long exact sequence, to ultimately recover the following expression for the Poisson cohomology of a general b^m-Poisson manifold:

$$\mathrm{H}^k_\Pi(M) \simeq \mathrm{H}^k(M) \oplus \mathrm{H}^{k-1}(Z) \oplus \left(\mathrm{H}^{k-1}_\Pi(\mathcal{F}_Z)\right)^{m-1} \oplus \left(\frac{\mathrm{H}^{k-1}(Z)}{\alpha \wedge \mathrm{H}^{k-1}(\mathcal{F}_Z)}\right)^{m-1}.$$

These results open several questions for future projects. For simple b^m-Poisson manifolds, where the computation of Poisson cohomology can be carried out explicitly, we have been able to measure the failure of the isomorphism in cohomology by appropriately unfolding the Poisson structure in terms of a real parameter ε. The integrating coboundaries in cohomology blow-up taking the limit $\varepsilon \to 0$, showing the analytic behaviour of the pathology. Can similar computations be carried out in general? Additionally, this example shows the cohomology of the symplectic algebroid might be very different from the cohomology of the resulting Poisson manifold. An interesting problem is to find general relations between both cohomologies in terms of the anchor map ρ.

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